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Stochastic stability of extended filtering for non-linear systems with measurement packet losses

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Abstract: This study is concerned with stochastic stability of a new extended filtering for non-linear systems subject to measurement packet losses. The measurements sensored are transmitted to the estimator through a packet-dropping network. By introducing a time-stamped packet arrival indicator sequence, the measurement loss process is modelled as an independent, identically distributed (i.i.d.) and therefore a Bernoulli process. The boundedness of estimation error covariance matrices is proved by showing the existence of a critical threshold for measurement packet arrival probability. It is also shown that, under appropriate assumptions, the estimation error remains bounded as long as the noise covariance matrices and the initial estimation error can be ensured small enough. Finally, simulation results validating the effectiveness of this proposed filtering framework are also presented.

1 Introduction

In recent years, considerable attention has been paid to networked control systems where the communication between sensors, controllers and actuators is often realised through a shared network medium [1]. Networked control systems possess many advantages over the traditional point-to-point control systems, such as low cost, reduced power requirements, easy installation, simple maintenance and high reliability [2]. However, transmitting the observed data and control signals across the network inevitably results in some newly emerged problems including, but not limited to, random time-delay, packet loss, quantisation and channel fading, which render the estimation and control for networked control systems significantly challenging [3–8]. In particular, the state estimation problem across a network has gained recurring research attention in the literature.

Pioneering works on linear estimation with measurement losses can be traced back to [9–11]. In [9], the packet loss process was modelled as an independent, identically distributed (i.i.d.) Bernoulli process and then the linear minimum mean-squared error (LMMSE) estimator with filter iterations similar to the standard Kalman filter by only utilising the statistics of packet arrival indicator sequence was derived. In [10], a sufficient and necessary condition for the existence of the linear recursive estimator when the packet loss sequence was not necessarily i.i.d. was established. Moreover, the condition for the uniformly asymptotic stability of the LMMSE estimator was developed for linear systems with measurements corrupted by white multiplicative noise in [11]. It is worth mentioning that in this case the error covariance matrix is governed by a deterministic equation iteration and therefore an equivalent linear system without measurement losses can be constructed to facilitate the asymptotic stability analysis of the LMMSE estimator.

Since widely application of the clairvoyant Kalman filter, ranging from tracking, detection to control, Kalman filtering with intermittent observations has attracted relative attention in last decade; see [12–14]. In [12], the effects of packet loss resulted from unreliability of the network on the stability and performance of Kalman filtering were investigated for Bernoulli packet loss process. In addition, inspired by the uncertainty threshold principle, it has also been shown there exists a critical value for the packet arrival probability such that the averaged estimation error covariance matrix will be bounded for any initial condition if the probability exceeds the critical probability; otherwise, the averaged estimation error covariance matrix will diverge for some initial conditions. To capture the possible transient correlation of network channel variations, the packet loss process was modelled as a time-homogenous two-state Markov chain in [13, 14] and sufficient conditions for the introduced peak covariance stability for general vector systems were established therein. In cases above, Kalman filtering with intermittent observations under different packet loss models is studied by only utilising the time-stamped packet loss indicator sequence, which renders the estimation error covariance matrix iteration stochastic and therefore poses significant challenges to its stability analysis. More recently, however, a suboptimal estimator for linear systems with Bernoulli packet losses was proposed in [15], by combining the use of time-stamped measurement innovation sequence and statistics of the indicator stochastic variable when designing the filter, to balance the performance and the stability analysis.

In the presence of non-linearities and packet losses, a multitude of publications can be widely found in the literature; see, for example, [16–23] and references therein. Among them, an extended minimum-variance filter was developed for non-linear stochastic systems in [16] and the recursive finite-horizon filter was proposed for a class of non-linear time-varying systems subject to multiplicative noises, measurement packet losses and quantisation effects in [19]. In [20], the distributed H_∞ filtering problem was well investigated for a wide class of non-linear systems with successive packet losses. Furthermore, the extended Kalman filtering with intermittent observations (called intermittent extended Kalman filter for short) and unscented Kalman filtering with intermittent observations (similarly, called intermittent unscented Kalman filter (IUKF) for short) have recently been studied in [21–23], respectively. Followed the spirit of [24], conditions for guaranteeing stochastic stability of the intermittent extended Kalman filter and the IUKF are developed, respectively. In addition, it has been shown in [22, 23] that the existence of a critical value for packet arrival probability to ensure statistical convergence of the averaged error covariance matrix.

The present paper, which can be seen as a further complementary to the prior works mentioned above, is concerned with the issue of estimating the state of a general non-linear system, where the state estimate is based on the observed data provided by an unreliable sensor network. The contributions of this paper include: similarly to the intermittent extended Kalman filter, we first extend the recently proposed linear suboptimal filter in [15] to non-linear case and derive a new extended filter (EF). We then generalise the concept of uniform observability to the case considered in this paper. Unlike the statistical convergence properties of averaged error covariance matrices for intermittent extended Kalman filter, the asymptotic convergence properties of the error covariance matrices can be derived for the proposed estimator under appropriate assumptions. Moreover, conditions for guaranteeing stochastic stability of estimation error of the proposed EF are also established. Specifically, whenever the non-linearities are not severe, simulation results show that the new EF has similar estimation performance with IUKF in [23] but with easier computations [25].

The remainder of this paper is organised as follows. We briefly introduce the non-linear system to be considered in Section 2 and establish the proposed EF in Section 3, which is followed by our main results on the boundedness of the error covariance matrices and the stochastic stability of the estimation error in Sections 4 and 5, respectively. A numerical example is presented in Section 6 and finally, the conclusions are drawn in Section 7.

Throughout this paper, the set of all non-negative integers is denoted by N; the Euclidean norm for real vectors or the spectral norm for real matrices is denoted by $\|\cdot\|$. Furthermore, We use $\boldsymbol{P} > \mathbf{0}$ $(\geq \mathbf{0})$ to represent the positive definite (positive semi-definite) matrix $\boldsymbol{P}$, use $\mathbb{E}\{\boldsymbol{x}\}$ to denote the expectation value of $\boldsymbol{x}$, and use C^1 to denote all the continuously differentiable functions.

2 System description and preliminaries

Consider the following non-linear discrete-time stochastic system with measurement packet losses

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k) + \boldsymbol{\omega}_k \tag{1}$$

$$\boldsymbol{y}_k = \gamma_k \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{\upsilon}_k \tag{2}$$

where $k = 0, 1, \ldots$ is the time instant, $\boldsymbol{x}_k \in \mathbb{R}^n$ is the state vector with the initial state $\boldsymbol{x}_0$, $\boldsymbol{y}_k \in \mathbb{R}^p$ is the measurement vector, and the process noise $\boldsymbol{\omega}_k \in \mathbb{R}^n$, measurement noise $\boldsymbol{\upsilon}_k \in \mathbb{R}^p$ are both zero-mean white Gaussian vectors with covariance matrices $\mathbb{E}\{\boldsymbol{\omega}_k \boldsymbol{\omega}_j^T\} = \boldsymbol{Q}_k \delta_{kj}$, $\mathbb{E}\{\boldsymbol{\upsilon}_k \boldsymbol{\upsilon}_j^T\} = \boldsymbol{R}_k \delta_{kj}$, respectively, where δ_{kj} is the Kronecker delta function. The initial state x_0 is also assumed to be a zero-mean white Gaussian random vector with covariance matrix $\mathbb{E}\{\boldsymbol{x}_0 \boldsymbol{x}_0^T\} = \boldsymbol{P}_0 > \mathbf{0}$. Moreover, the non-linear functions $\boldsymbol{f}, \boldsymbol{h}$ are assumed to be C^1 functions. For clarity, system (1) is considered autonomous, however, the results presented in this paper can be readily generalised to the controlled non-linear systems.

The packet arrival indicator variable γ_k is assumed to take binary values on 0 and 1. More precisely, $\gamma_k = 1$ represents that the measurement packet arrives at the remote estimator; however, $\gamma_k = 0$ indicates that the measurement packet drops in the network. Moreover, the random process is characterised by parameter λ with

$$\Pr\{\gamma_k = 1\} = \lambda \tag{3}$$

$$\Pr\{\gamma_k = 0\} = 1 - \lambda \tag{4}$$

where the packet arrival probability $\lambda \in [0, 1]$ is known, and the sequence γ_k, the noise process $\boldsymbol{\omega}_k, \boldsymbol{\upsilon}_k$ and the initial state $\boldsymbol{x}_0$ are assumed to be mutually independent for all $k \in N$.

Before moving on, the following auxiliary lemmas are introduced.

Lemma 1 [22]: Suppose that $\boldsymbol{UCU}^T < \boldsymbol{A}$ holds for matrix $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ and symmetric positive definite matrices $\boldsymbol{A}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$, then $\boldsymbol{U}^T\boldsymbol{A}^{-1}\boldsymbol{U} < \boldsymbol{C}^{-1}$ holds.

Lemma 2 [22]: For symmetric positive-definite matrices $\boldsymbol{A}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$, $(\boldsymbol{A}+\boldsymbol{C})^{-1} > \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{C}\boldsymbol{A}^{-1}$ holds.

3 Derivation of the EF

In this section, the EF for general non-linear systems with measurement packet losses will be constructed. To proceed, the following assumption on packet arrival indicator sequence is stated.

Assumption 1: γ_k is supposed to be time-stamped, that is, the value of γ_k at every time instant k can be observed. So the information γ_k together with observation $\boldsymbol{y}_k$ are available in the estimator design.

Linearise non-linear functions $\boldsymbol{f}$ and $\boldsymbol{h}$ at points $\hat{\boldsymbol{x}}_{k|k}, \hat{\boldsymbol{x}}_{k|k-1}$, respectively.

$$\boldsymbol{f}(\boldsymbol{x}_k) = \boldsymbol{f}(\hat{\boldsymbol{x}}_{k|k}) + \boldsymbol{F}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k}) + \boldsymbol{\varphi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k})$$

$$\boldsymbol{h}(\boldsymbol{x}_k) = \boldsymbol{h}(\hat{\boldsymbol{x}}_{k|k-1}) + \boldsymbol{H}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1}) + \boldsymbol{\psi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k-1})$$

where $\boldsymbol{F}_k = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{\hat{\boldsymbol{x}}_{k|k}}$ and $\boldsymbol{H}_k = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\Big|_{\hat{\boldsymbol{x}}_{k|k-1}}$, denote the Jacobian matrices of non-linear functions $\boldsymbol{f}, \boldsymbol{h}$ at points $\hat{\boldsymbol{x}}_{k|k}, \hat{\boldsymbol{x}}_{k|k-1}$, respectively. Then the linearised approximation of the original non-linear system becomes

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{F}_k\boldsymbol{x}_k + \boldsymbol{\omega}_k + \left[\boldsymbol{f}(\hat{\boldsymbol{x}}_{k|k}) - \boldsymbol{F}_k\hat{\boldsymbol{x}}_{k|k} + \boldsymbol{\varphi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k})\right] \\ \boldsymbol{y}_k &= \gamma_k\boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{\upsilon}_k \\ &\quad + \left[\gamma_k\boldsymbol{h}(\hat{\boldsymbol{x}}_{k|k-1}) - \gamma_k\boldsymbol{H}_k\hat{\boldsymbol{x}}_{k|k-1} + \gamma_k\boldsymbol{\psi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k-1})\right]\end{aligned}$$

For the following linear discrete-time stochastic system with measurement packet losses

$$x_{k+1} = Fx_k + \omega_k$$
$$y_k = \gamma_k H x_k + v_k$$

where F and H are, respectively, the state transition matrix and the observation matrix. Then the linear suboptimal filter developed for the above system in [14] is specified as

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + \lambda P_{k|k-1}H^T(\lambda H P_{k|k-1}H^T + R)^{-1} \\ &\quad \times (y_k - \gamma_k H\hat{x}_{k|k-1}) \\ \hat{x}_{k+1|k} &= F\hat{x}_{k|k} \\ P_{k|k} &= P_{k|k-1} - \lambda^2 P_{k|k-1}H^T(\lambda H P_{k|k-1}H^T + R)^{-1}HP_{k|k-1} \\ P_{k+1|k} &= FP_{k|k}F^T + Q\end{aligned}$$

So, the proposed EF can be stated as follows

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + \lambda P_{k|k-1}H_k^T(\lambda H_k P_{k|k-1}H_k^T + R_k)^{-1} \\ &\quad \times (y_k - \gamma_k h(\hat{x}_{k|k-1})) \qquad (5)\\ \hat{x}_{k+1|k} &= f(\hat{x}_{k|k}) \qquad (6)\\ P_{k|k} &= P_{k|k-1} - \lambda^2 P_{k|k-1}H_k^T \\ &\quad \times (\lambda H_k P_{k|k-1}H_k^T + R_k)^{-1}H_k P_{k|k-1} \qquad (7)\\ P_{k+1|k} &= F_k P_{k|k}F_k^T + Q_k \qquad (8)\end{aligned}$$

Remark 1: As pointed out in Section 1, the intermittent extended Kalman filter iteration is very much involved with the stochastic packet arrival indicator variable sequence $\{\gamma_k\}_0^\infty$, so the intermittent extended Kalman filter iteration is inherently stochastic and therefore can only be determined online. Moreover, only statistical properties on averaged estimation error covariance matrices can be derived for intermittent extended Kalman filter. Note, however that the EF iteration devised in this paper (see (7)–(8)) is readily expressed in terms of expectation value of random variable γ_k, is deterministic and therefore can be determined offline, which can be viewed as one characteristic property of our proposed estimator. Nevertheless, it should be clearly clarified that the EF in this paper is of worse estimation performance than the intermittent extended Kalman filter because it makes a tradeoff between the estimation performance and the offline calculation capability.

4 Boundedness of the estimation error covariance matrices

The results in the proceeding section show that the estimation error covariance matrices of the discrete-time EF stated by (5) to (8) remain bounded, if the non-linear system considered is posited to satisfy appropriate conditions. These conditions include requirements of boundedness of the system parameter matrices and the existence of the invertible matrix of observable matrix. Before continuing on, a generalised concept in terms of the uniform observability of the EF is first introduced.

Consider the following linear time-varying system with missing measurement packet losses

$$x_{k+1} = F_k x_k + \omega_k \qquad (9)$$
$$y_k = \gamma_k H_k x_k + v_k \qquad (10)$$

where system parameter matrices F_k, H_k depend on the estimates $\hat{x}_{k|k}, \hat{x}_{k|k-1}$, respectively, and Bernoulli random variable γ_k is time-stamped with $\mathbb{E}\{\gamma_k\} = \lambda$. Similar to [21], the concept of uniform observability for linear time-varying systems can also be extended to systems with measurement packet losses discussed in this paper.

Definition 1: Let the generalised observability Gramian be given by

$$\check{M}_{k+s,k} = \sum_{i=k}^{k+s} \lambda \Phi_{i,k}^T(\sqrt{\lambda}H_i)^{\mathrm{T}}(\sqrt{\lambda}H_i)\Phi_{i,k} \qquad (11)$$

where the transition matrix $\Phi_{i,k} = F_i F_{i+1}\cdots F_k$ with $\Phi_{i,i} = I_n$. Then the matrix pair (F_k, H_k) is said to be uniformly observable if there exist some integer $s > 0$ and two positive real constants $\underline{m}, \overline{m} > 0$, such that the generalised observability Gramian $\check{M}_{k+s,k}$ satisfies

$$\mathbf{0} < \underline{m}I_n \le \check{M}_{k+s,k} \le \overline{m}I_n \qquad (12)$$

for every $k \in N$.

Assumption 2: There exist positive real constants $\overline{f}, \underline{h}, \overline{h}, \underline{q}, \overline{q}, \underline{r}, \overline{r}$, such that the following bounds on system parameter matrices hold for every $k \in N$

$$\|F_k\| \le \overline{f} \qquad (13)$$
$$\underline{h}^2 \le \|H_k^T H_k\| \le \overline{h}^2 \qquad (14)$$
$$\underline{q}I_n \le Q_k \le \overline{q}I_n \qquad (15)$$
$$\underline{r}I_p \le R_k \le \overline{r}I_p \qquad (16)$$

Remark 2: Observe, that if $\overline{f} < 1$, the desired bounds for the error covariance matrices (17) subject to (7)–(8) can be derived directly from Assumption 2, even for the worst case $\lambda = 0$, that is, the desired bounds are independent of the measurement process if $\overline{f} < 1$.

In the following, the Bernoulli measurement packet loss process with a non-zero parameter λ will be discussed and the bounds for the error covariance matrices will be derived.

Theorem 1: Under Assumption 2 and assume that (F_k, H_k) is uniformly observable, that H_k^{-1} exists for every $k \in N$, then there exists a critical value for packet arrival probability $\lambda_c = 1 - \frac{1}{\overline{f}^2}$ such that the error covariance matrices are bounded provided that $\lambda > \lambda_c$, that is, there exists a positive real constant pair $\underline{p}, \overline{p}$, such that

$$\underline{p}I_n \le P_{k+1|k+1} \le P_{k+1|k} \le \overline{p}I_n \qquad (17)$$

Proof: Since H_k is invertible for every $k \in N$, given that (F_k, H_k) is uniformly observable and thereby detectable, even for $\lambda = 1$, the lower bound can be obtained directly from [26]. Clearly, according to (7)–(8), we have

$$\begin{aligned}P_{k+1|k} &= F_k\Big[P_{k|k-1} - \lambda^2 P_{k|k-1}H_k^T \\ &\quad \times \left(\lambda H_k P_{k|k-1}H_k^T + R_k\right)^{-1} H_k P_{k|k-1}\Big]F_k^T + Q_k\end{aligned}$$

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Setting $\boldsymbol{A} = \lambda \boldsymbol{H}_k \boldsymbol{P}_{k|k-1} \boldsymbol{H}_k^T, \boldsymbol{C} = \boldsymbol{R}_k$ and using Lemma 2 to the inverse term above yields

$$\boldsymbol{P}_{k+1|k} \le (1-\lambda)\boldsymbol{F}_k \boldsymbol{P}_{k|k-1} \boldsymbol{F}_k^T + \boldsymbol{F}_k \boldsymbol{H}_k^{-1} \boldsymbol{R}_k \boldsymbol{H}_k^{-T} \boldsymbol{F}_k^T + \boldsymbol{Q}_k$$

Considering the bounds on matrices $\boldsymbol{H}_k, \boldsymbol{R}_k, \boldsymbol{Q}_k$, we have

$$\begin{aligned} \boldsymbol{P}_{k+1|k} &\le (1-\lambda)\boldsymbol{F}_k \boldsymbol{P}_{k|k-1} \boldsymbol{F}_k^T + \frac{\overline{r}}{\underline{h}^2} \boldsymbol{F}_k \boldsymbol{F}_k^T + \overline{q} \boldsymbol{I}_n \\ &\le (1-\lambda)\overline{f}^2 \boldsymbol{P}_{k|k-1} + \left(\frac{\overline{r}\overline{f}^2}{\underline{h}^2} + \overline{q} \right) \boldsymbol{I}_n \end{aligned}$$

Recursively, it follows that

$$\begin{aligned} \boldsymbol{P}_{k+1|k} &\le \left[(1-\lambda)\overline{f}^2\right]^k \boldsymbol{P}_{1|0} + \left(\frac{\overline{r}\overline{f}^2}{\underline{h}^2} + \overline{q} \right) \sum_{i=0}^{k-1} \left[(1-\lambda)\overline{f}^2\right]^i \boldsymbol{I}_n \\ &\le \left\{ \left[(1-\lambda)\overline{f}^2\right]^k \|\boldsymbol{P}_{1|0}\| + \left(\frac{\overline{r}\overline{f}^2}{\underline{h}^2} + \overline{q} \right) \sum_{i=0}^{k-1} \left[(1-\lambda)\overline{f}^2\right]^i \right\} \\ &\quad \times \boldsymbol{I}_n \end{aligned} \tag{18}$$

Denote $\tilde{p} = \max \left\{ \|\boldsymbol{P}_{1|0}\|, \left(\frac{\overline{r}\overline{f}^2}{\underline{h}^2} + \overline{q} \right) \right\}$ and rewrite (18) as follows

$$\boldsymbol{P}_{k+1|k} \le \tilde{p} \sum_{i=0}^{k} \left[(1-\lambda)\overline{f}^2\right]^i \boldsymbol{I}_n, \quad \forall k \ge 0 \tag{19}$$

It is noteworthy that under the assumption $\lambda > 1 - \frac{1}{\overline{f}^2}$, the sum in (19) converges to $\overline{p} = \frac{\tilde{p}}{1-(1-\lambda)\overline{f}^2}$ and therefore the upper bound in (17) follows directly from (19). □

Remark 3: It should be noted that, similarly to the intermittent extended Kalman filter for general non-linear systems, Riccati-like iteration of estimation error covariance $\boldsymbol{P}_{k|k-1}$ can be just seen as a first-order approximation to the true error covariance, which does not generally possess a linear iteration. Therefore boundedness of error covariances $\boldsymbol{P}_{k|k-1}, \boldsymbol{P}_{k|k}$ do not necessarily imply stochastic stability of the estimation error $\boldsymbol{e}_{k|k-1}$. Then study of the behaviour of the averaged estimation error $\mathbb{E}\{\boldsymbol{e}_{k|k-1}\}$ in Section 5 are of significant implications.

5 Stochastic stability of the estimation error

In this section, the estimation error resulted from the proposed EF will be shown to be bounded, if some appropriate assumptions hold.

Assumption 3: There exist positive real constants $\overline{f}, \underline{f}, \overline{h}, \overline{q}, \underline{q}, \overline{r}, \underline{r}$ such that the following bounds on system parameter matrices hold for every $k \in N$

$$\underline{f}^2 \le \|\boldsymbol{F}_k^T \boldsymbol{F}_k\| \le \overline{f}^2 \tag{20}$$

$$\|\boldsymbol{H}_k\| \le \overline{h} \tag{21}$$

$$\underline{q}\boldsymbol{I}_n \le \boldsymbol{Q}_k \le \overline{q}\boldsymbol{I}_n \tag{22}$$

$$\underline{r}\boldsymbol{I}_p \le \boldsymbol{R}_k \le \overline{r}\boldsymbol{I}_p \tag{23}$$

Assumption 4: For any positive real constant pair $\epsilon_\varphi, \epsilon_\psi$, there exists another positive real constant pair $\delta_\varphi, \delta_\psi$ such that the following two inequalities hold for all $\|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k}\| \le \delta_\varphi$ and $\|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1}\| \le \delta_\psi$, respectively

$$\|\boldsymbol{\varphi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k})\| \le \epsilon_\varphi \|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k}\|^2 \tag{24}$$

$$\|\boldsymbol{\psi}(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k-1})\| \le \epsilon_\psi \|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1}\|^2 \tag{25}$$

Assumption 5: There exist two positive real constants $\overline{p}, \underline{p}$ such that the error covariances $\boldsymbol{P}_{k|k-1}, \boldsymbol{P}_{k|k}$ are bounded, that is

$$\underline{p}\boldsymbol{I}_n \le \|\boldsymbol{P}_{k|k}\| \le \|\boldsymbol{P}_{k|k-1}\| \le \overline{p}\boldsymbol{I}_n \tag{26}$$

Remark 4: Note that Theorem 1 derived explicit condition for packet arrival probability λ to ensure the boundedness of estimation error covariance matrices for systems where $\boldsymbol{H}_k^{-1}$ exists. However, for general time-varying systems no explicit conditions of λ have been given in the literature. To lessen this strict requirement, similarly to [22, 23], we assume the boundedness of estimation error covariance matrices in Assumption 5.

Lemma 3: Let $g(\lambda)$ be

$$g(\lambda) = (1+\theta)^{-1}[1 + \delta(\lambda - \lambda^2)]$$

where constants $\theta > 0, \delta > 0$. Then there always exists a set of the form $S = (\lambda_n, 1]$ such that $g(\lambda) < 1$ holds for all $\lambda \in S$.

Proof: Note clearly, that $g(\lambda) < 1$ holds at least for both $\lambda = 1$ and 0. However, for $\lambda = 0$, the assumption on the boundedness of the error covariance matrices $\boldsymbol{P}_{k+1|k+1}$ and $\boldsymbol{P}_{k+1|k}$ can be easily violated except the very case $\overline{g} < 1$. Moreover, $g(\lambda)$ is continuous over $[0, 1]$ and reaches its maximum value at point $\lambda = 1/2$. From $\lambda = 1/2$ to $\lambda = 1, g(\lambda)$ decreases. Considering $g(1) = (1+\theta)^{-1} < 1$ and the continuity of $g(\lambda)$, then there always exists a neighbourhood of point $\lambda = 1$, denoted by $S = (\lambda_n, 1]$, in which $g(\lambda) < 1$ holds. □

Theorem 2: Consider the non-linear stochastic system described by (1)–(2) and an EF given by (5)–(8). Under Assumptions 3–5 and that $\lambda \in S$, if there exists a real constant $\epsilon > 0$, such that $\mathbb{E}\{\|\boldsymbol{e}_{1|0}\|^2\} \le \epsilon$, then the state estimation error $\boldsymbol{e}_{k|k-1}$ as stated in (27) is exponentially bounded in mean square and bounded with probability one.

Proof: According to Assumption 5, there exists positive real constants $\overline{p}, \underline{p}$ such that $\underline{p}\boldsymbol{I}_n \le \|\boldsymbol{P}_{k|k}\| \le \|\boldsymbol{P}_{k|k-1}\| \le \overline{p}\boldsymbol{I}_n$.

From (5)–(6), we have the state estimation error defined by

$$\begin{aligned} \boldsymbol{e}_{k+1|k} &= \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k} \\ &= \boldsymbol{F}_k(\boldsymbol{I}_n - \gamma_k \boldsymbol{K}_k)\, \boldsymbol{e}_{k|k-1} + \boldsymbol{r}_k + \boldsymbol{s}_k \end{aligned} \tag{27}$$

where, we define $\boldsymbol{r}_k = \boldsymbol{\varphi}_k - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{\psi}_k$ and $\boldsymbol{s}_k = \boldsymbol{\omega}_k - \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{\upsilon}_k$. Define $V_k(\boldsymbol{e}_{k|k-1}) = \boldsymbol{e}_{k|k-1}^T \boldsymbol{P}_{k|k-1}^{-1} \boldsymbol{e}_{k|k-1}$, then

$$\begin{aligned} V_{k+1}(\boldsymbol{e}_{k+1|k}) &= \boldsymbol{e}_{k+1|k}^T \boldsymbol{P}_{k+1|k}^{-1} \boldsymbol{e}_{k+1|k} \\ &= \left[\boldsymbol{F}_k(\boldsymbol{I}_n - \gamma_k \boldsymbol{F}_k \boldsymbol{H}_k)\boldsymbol{e}_{k|k-1} + \boldsymbol{s}_k + \boldsymbol{r}_k\right]^T \\ &\quad \times \boldsymbol{P}_{k+1|k}^{-1} \left[\boldsymbol{F}_k(\boldsymbol{I} - \gamma_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{e}_{k|k-1} + \boldsymbol{s}_k + \boldsymbol{r}_k\right] \end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{e}_{k|k-1}^T(\boldsymbol{I}_n - \gamma_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{F}_k^T \boldsymbol{P}_{k+1|k}^{-1} \\
&\quad \times \boldsymbol{F}_k(\boldsymbol{I} - \gamma_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{e}_{k|k-1} \\
&\quad + \boldsymbol{s}_k^T \boldsymbol{P}_{k+1|k}^{-1}\left[2\boldsymbol{F}_k(\boldsymbol{I}_n - \gamma_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{e}_{k|k-1} + 2\boldsymbol{r}_k\right] \\
&\quad + \boldsymbol{r}_k^T \boldsymbol{P}_{k+1|k}^{-1}\left[2\boldsymbol{F}_k(\boldsymbol{I}_n - \gamma_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{e}_{k|k-1} + \boldsymbol{r}_k\right] \\
&\quad + \boldsymbol{s}_k^T \boldsymbol{P}_{k+1|k}^{-1}\boldsymbol{s}_k
\end{aligned} \tag{28}
$$

For the first term above, it yields

$$
\begin{aligned}
&\mathbb{E}\left\{(\boldsymbol{F}_k - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F} - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)\right\} \\
&\quad = (\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k) \\
&\qquad + (\lambda - \lambda^2)(\boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k) \\
&\quad \le \left[1 + \frac{(\lambda - \lambda^2)\bar{p}\bar{f}^2\bar{k}^2\bar{h}^2}{\underline{p}\underline{f}^2\underline{d}^2}\right] \\
&\qquad \times (\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{F}_k^T \boldsymbol{P}_{k+1|k}^{-1} \boldsymbol{F}_k (\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k)
\end{aligned} \tag{29}
$$

where we denote by $\bar{k}, \underline{d}$ the upper bound on the matrix norm $\|\boldsymbol{K}_k\|$ and the lower bound on the matrix norm $\|\boldsymbol{D}_k\| = \|\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k\|$, respectively, which are calculated as follows. Since $\boldsymbol{K}_k = \lambda \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T(\lambda \boldsymbol{H}_k \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T + \boldsymbol{R}_k)^{-1}$ and (20)–(23), considering $\lambda \boldsymbol{H}_k \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T \ge 0$, it easily follows that

$$
\|\boldsymbol{K}_k\| \le \frac{\lambda \bar{p}\bar{h}}{\underline{r}} \le \frac{\bar{p}\bar{h}}{\underline{r}} \triangleq \bar{k} \tag{30}
$$

Meanwhile, $\underline{d}$ can be determined in the following. In (7), it states that

$$
\begin{aligned}
&(\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{P}_{k|k-1} \\
&\quad = \boldsymbol{P}_{k|k-1} - \lambda^2 \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T(\lambda \boldsymbol{H}_k \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T + \boldsymbol{R}_k)^{-1}\boldsymbol{H}_k \boldsymbol{P}_{k|k-1} \\
&\quad = (1-\lambda)\boldsymbol{P}_{k|k-1} + \lambda\Big[\boldsymbol{P}_{k|k-1} - \boldsymbol{P}_{k|k-1}(\sqrt{\lambda}\boldsymbol{H}_k)^T \\
&\qquad \times \left(\sqrt{\lambda}\boldsymbol{H}_k \boldsymbol{P}_{k|k-1}\sqrt{\lambda}\boldsymbol{H}_k^T + \boldsymbol{R}_k\right)^{-1}(\sqrt{\lambda}\boldsymbol{H}_k)\boldsymbol{P}_{k|k-1}\Big]
\end{aligned} \tag{31}
$$

then using the matrix inversion lemma [27] to the second term in last equation yields

$$
\begin{aligned}
\boldsymbol{P}_{k|k} &= (\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{P}_{k|k-1} \\
&= (1-\lambda)\boldsymbol{P}_{k|k-1} + \lambda\left(\boldsymbol{P}_{k|k-1}^{-1} + \lambda \boldsymbol{H}_k^T \boldsymbol{R}_k^{-1}\boldsymbol{H}_k\right)^{-1}
\end{aligned} \tag{32}
$$

Owing to the positive definiteness, therefore the invertibility of covariance matrices $\boldsymbol{P}_{k|k}, \boldsymbol{P}_{k|k-1}$, it can be obtained that

$$
\begin{aligned}
&\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k \\
&\quad = \left[(1-\lambda)\boldsymbol{P}_{k|k-1} + \lambda\left(\boldsymbol{P}_{k|k-1}^{-1} + \lambda \boldsymbol{H}_k^T \boldsymbol{R}_k^{-1}\boldsymbol{H}_k\right)^{-1}\right]\boldsymbol{P}_{k|k-1}^{-1}
\end{aligned}
$$

where, $\boldsymbol{P}_{k|k-1} > \boldsymbol{0}, (\boldsymbol{P}_{k|k-1}^{-1} + \lambda \boldsymbol{H}_k^T \boldsymbol{R}_k^{-1}\boldsymbol{H}_k)^{-1} > \boldsymbol{0}$, so $[(1-\lambda)\boldsymbol{P}_{k|k-1} + \lambda(\boldsymbol{P}_{k|k-1}^{-1} + \lambda \boldsymbol{H}_k^T \boldsymbol{R}_k^{-1}\boldsymbol{H}_k)^{-1}] > \boldsymbol{0}, \boldsymbol{P}_{k|k-1}^{-1} > \boldsymbol{0}$. According to matrix theory on eigenvalue and singular value estimates of product of two positive-definite matrices; see [28, Theorem 3.1], it yields

$$
\begin{aligned}
&\|\boldsymbol{I}_n - \lambda \boldsymbol{K}_k \boldsymbol{H}_k\| \\
&\quad = \left\|\left[(1-\lambda)\boldsymbol{P}_{k|k-1} + \lambda\left(\boldsymbol{P}_{k|k-1}^{-1} + \lambda \boldsymbol{H}_k^T \boldsymbol{R}_k^{-1}\boldsymbol{H}_k\right)^{-1}\right]\boldsymbol{P}_{k|k-1}^{-1}\right\| \\
&\quad \ge \frac{\underline{p}}{\bar{p}}\left(1 - \lambda + \frac{\lambda \underline{r}}{\underline{r} + \lambda \bar{h}^2 \underline{p}}\right) \ge \frac{\underline{p}\underline{r}}{\bar{p}(\underline{r} + \bar{h}^2 \underline{p})} \triangleq \underline{d}
\end{aligned} \tag{33}
$$

On the other hand, from (8), it follows

$$
\boldsymbol{P}_{k+1|k} = \boldsymbol{F}_k \boldsymbol{P}_{k|k}\boldsymbol{F}_k^T + \boldsymbol{Q}_k \ge \left(1 + \frac{\underline{q}}{\bar{p}\bar{f}^2}\right)\boldsymbol{F}_k \boldsymbol{P}_{k|k}\boldsymbol{F}_k^T
$$

and now substituting the last equation into (7) yields

$$
\begin{aligned}
\boldsymbol{P}_{k+1|k} &\ge \left(1 + \frac{\underline{q}}{\bar{p}\bar{f}^2}\right)\Big[\boldsymbol{F}_k \boldsymbol{P}_{k|k-1}\boldsymbol{F}_k^T - \lambda^2 \boldsymbol{F}_k \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T \\
&\quad \times (\lambda \boldsymbol{H}_k \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T + \boldsymbol{R}_k)^{-1}\boldsymbol{H}_k \boldsymbol{P}_{k|k-1}\boldsymbol{F}_k^T\Big] \\
&\ge \left(1 + \frac{\underline{q}}{\bar{p}\bar{f}^2}\right)\Big[(\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{P}_{k|k-1} \\
&\quad \times (\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T + (\lambda - \lambda^2)\boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k \boldsymbol{P}_{k|k-1} \\
&\quad \times (\boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T + \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{R}_k(\boldsymbol{F}_k \boldsymbol{K}_k)^T\Big] \\
&\ge \left(1 + \frac{\underline{q}}{\bar{p}\bar{f}^2}\right)(\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{P}_{k|k-1} \\
&\quad \times (\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T
\end{aligned} \tag{34}
$$

where, similarly, $(\lambda - \lambda^2)\boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k \boldsymbol{P}_{k|k-1}(\boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T + \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{R}_k(\boldsymbol{F}_k \boldsymbol{K}_k)^T \ge \boldsymbol{0}$. Set $\boldsymbol{A} = \boldsymbol{P}_{k+1|k}, \boldsymbol{U} = \boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k$, $\boldsymbol{C} = \left(1 + \frac{q}{\bar{p}\bar{f}^2}\right)\boldsymbol{P}_{k|k-1}$ in (34) and use Lemma 1 to show that the following inequality holds

$$
\begin{aligned}
&(\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k) \\
&\quad < \left(1 + \frac{\underline{q}}{\bar{p}\bar{f}^2}\right)^{-1}\boldsymbol{P}_{k|k-1}^{-1}
\end{aligned} \tag{35}
$$

Combining (29) and (35) yields

$$
\begin{aligned}
&\mathbb{E}\left\{(\boldsymbol{F}_k - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F} - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)\right\} \\
&\quad \le \left[1 + \frac{(\lambda - \lambda^2)\bar{p}\bar{f}^2\bar{k}^2\bar{h}^2}{\underline{p}\underline{f}^2\underline{d}^2}\right] \\
&\qquad \times (\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F}_k - \lambda \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k) \\
&\quad < \left(1 + \frac{\underline{q}}{\bar{p}\bar{f}^2}\right)^{-1}\left[1 + \frac{(\lambda - \lambda^2)\bar{p}\bar{f}^2\bar{k}^2\bar{h}^2}{\underline{p}\underline{f}^2\underline{d}^2}\right]\boldsymbol{P}_{k|k-1}^{-1}
\end{aligned}
$$

In Lemma 3, by letting $\theta = \frac{\underline{q}}{\bar{p}\bar{f}^2} > 0, \delta = \frac{\bar{p}\bar{f}^2\bar{k}^2\bar{h}^2}{\underline{p}\underline{f}^2\underline{d}^2} > 0$, it follows $g(\lambda) < 1$ for all $\lambda \in S$. Hence, there exists a real number $0 < \alpha < 1$, such that $g(\lambda) = 1 - \alpha$ holds for $\lambda \in S$. Therefore we can derive

$$
\begin{aligned}
&\mathbb{E}\left\{(\boldsymbol{F}_k - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)^T \boldsymbol{P}_{k+1|k}^{-1}(\boldsymbol{F}_k - \gamma_k \boldsymbol{F}_k \boldsymbol{K}_k \boldsymbol{H}_k)\right\} \\
&\quad < (1-\alpha)\boldsymbol{P}_{k|k-1}^{-1}
\end{aligned} \tag{36}
$$

Note, also that the expectation value of the second term $\mathbb{E}\{\boldsymbol{s}_k^T \boldsymbol{P}_{k+1|k}^{-1} \times [2\boldsymbol{F}_k(\boldsymbol{I}_n - \gamma_k \boldsymbol{K}_k \boldsymbol{H}_k)\boldsymbol{e}_{k|k-1} + 2\boldsymbol{r}_k]\}$ in (28)

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becomes zero owing to zero-mean Gaussian noises ω_k, υ_k in the term s_k and mutual uncorrelation between the noises and the sequence γ_k.

The last two terms in (28), that is, $r_k^T P_{k+1|k}^{-1}[2F_k(I_n - \gamma_k K_k H_k)e_{k|k-1} + r_k]$ and $s_k^T P_{k+1|k}^{-1} s_k$ can be shown to be bounded in mean square sense as follows. Since $\|\gamma_k\| \le 1$, then

$$\|r_k\| = \|\varphi_k - \gamma_k F_k K_k \psi_k\| \le \|\varphi_k\| + \|\gamma_k F_k K_k \psi_k\|$$
$$\le \left(\epsilon_\varphi + \frac{\bar{f}\bar{p}\bar{h}}{\underline{r}}\epsilon_\psi\right)\|x_k - \hat{x}_{k|k-1}\|^2$$

holds for $\|x_k - \hat{x}_{k|k-1}\| \le \delta_\psi$, that is, with $\epsilon' = \left(\epsilon_\varphi + \frac{\bar{f}\bar{p}\bar{h}}{\underline{r}}\epsilon_\psi\right)$

$$\|r_k\| \le \epsilon'\|x_k - \hat{x}_{k|k-1}\|^2$$

Then for $\|x_k - \hat{x}_{k|k-1}\| \le \delta_\psi$, it follows that

$$r_k^T P_{k+1|k}^{-1}[2F_k(I_n - \gamma_k K_k H_k)e_{k|k-1} + r_k] \le \epsilon'\|x_k - \hat{x}_{k|k-1}\|^2 \frac{1}{\underline{p}}$$
$$\times\left[2\bar{f}\left(1 + \frac{\bar{p}\bar{h}^2}{\underline{r}}\right)\|x_k - \hat{x}_{k|k-1}\| + \epsilon'\delta_\psi\|x_k - \hat{x}_{k|k-1}\|\right] \quad (37)$$

By letting $\rho = \epsilon'\frac{1}{\underline{p}}\left[2\bar{f}\left(1 + \frac{\bar{p}\bar{h}^2}{\underline{r}}\right) + \epsilon'\delta_\psi\right]$, (37) reduces to

$$r_k^T P_{k+1|k}^{-1}\left[2F_k(I_n - \gamma_k K_k H_k)e_{k|k-1} + r_k\right] \le \rho\|x_k - \hat{x}_{k|k-1}\|^3$$

The second term $s_k^T P_{k+1|k}^{-1} s_k$ has been shown bounded in Lemma 3.3 in [24], that is, $\mathbb{E}\{s_k^T P_{k+1|k}^{-1} s_k\} \le \varrho$. Then taking the expectation value from both sides for the two terms above yields

$$\mathbb{E}\left\{r_k^T P_{k+1|k}^{-1}\left[2F_k(I_n - \gamma_k K_k H_k)e_{k|k-1} + r_k\right] + s_k^T P_{k+1|k}^{-1} s_k\right\}$$
$$\le \rho\mathbb{E}\left\{\|e_{k|k-1}\|^3\right\} + \varrho \quad (38)$$

Then from (36) and (38) above, it can be readily obtained that

$$\mathbb{E}\left\{V_{k+1}(e_{k+1|k})\right\} < (1-\alpha)\mathbb{E}\left\{V_k(e_{k|k-1})\right\}$$
$$+ \rho\mathbb{E}\left\{\|e_{k|k-1}\|^3\right\} + \varrho$$

for two real constants $\rho, \varrho > 0$ depending on parameters $\delta_\varphi, \delta_\psi, \epsilon_\varphi, \epsilon_\psi$ and the bounds of system parameter matrices as stated in (20)–(23). Then it is readily safe to draw the conclusion. □

Combing the results in Theorems 1 and 2, the following corollary can be readily stated.

Corollary 1: Consider the non-linear stochastic system described by (1)–(2) and an EF given by (5)–(8). Under Assumption 3–4 and assumption that H_k^{-1} exists for every $k \in N$, that (F_k, H_k) is uniformly observable, and that $\max\{\lambda_c, \lambda_n\} < \lambda \le 1$, if there exists a real constant $\epsilon > 0$, such that $\mathbb{E}\{\|e_{1|0}\|^2\} \le \epsilon$, then the state estimation error $e_{k|k-1}$ as stated in (27) is exponentially bounded in mean square and bounded with probability one.

Remark 5: Since the assumptions stated above satisfy those in Theorem 1, then based on Theorem 1, it can be concluded that there exist two positive real constants $\underline{p}, \bar{p}$, such that $\underline{p}I_n \le P_{k+1|k+1} \le P_{k+1|k} \le \bar{p}I_n$. The remaining proof follows from that of Theorem 2.

Remark 6: Clearly, this work can be viewed as a direct extension of that in [14] to the non-linear case and besides the basic idea of linearisation of the non-linear function, the novelties in the method also lie in that: (i) the covariance matrix iteration can be computed offline; (ii) the uniform observability can be quite different than the cases considered in the literature and therefore, the concept of uniform observability is largely modified; and (iii) comparing with that of the intermittent extended Kalman filter in [22], the stability analysis in this paper becomes much more complicated since the stochastic variable in [22] can be absorbed into the observation matrix and this reduces and simplifies the stability analysis of intermittent extended Kalman filter to that of the extended Kalman filter in [24].

6 Illustrative example

Consider the non-linear discrete-time system [23, 24]

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + \tau x_{2,k} \\ x_{2,k} + \tau\left[-x_{1,k} + (x_{1,k}^2 + x_{2,k}^2 - 1)\right] \end{bmatrix} + \omega_k \quad (39)$$

$$y_k = \gamma_k x_{1,k} + \upsilon_k \quad (40)$$

where $\tau = 10^{-3}$ and $\mathbb{E}\{\gamma_k\} = 0.14$. The covariances of ω_k and υ_k are $Q_k = 0.003^2 I_2$ and $R_k = 0.001^2$, respectively. The initial conditions are $x_{1,0} = 0.8, x_{2,0} = 0.2, \hat{x}_{1,0} = 2.3, \hat{x}_{2,0} = 2.2$ with $P_0 = I_2$. The simulation results were presented to compare the performance of the proposed EF and the IUKF in [23] in Figs. 1–2. For a fair comparison, we make a Monte Carlo test based on 50 independent samples and we use the root-mean-squared error (RMSE) of the second state component $x_{2,k}$. Besides the RMS time histories of the error metrics, another important evaluation metric is the required computation time of each filter. The mean computation times per filtering method (for the Monte Carlo test) run in Matlab 2011b on a 3.3-GHz, two-core Windows workstation are presented in Table 1. Our results show the

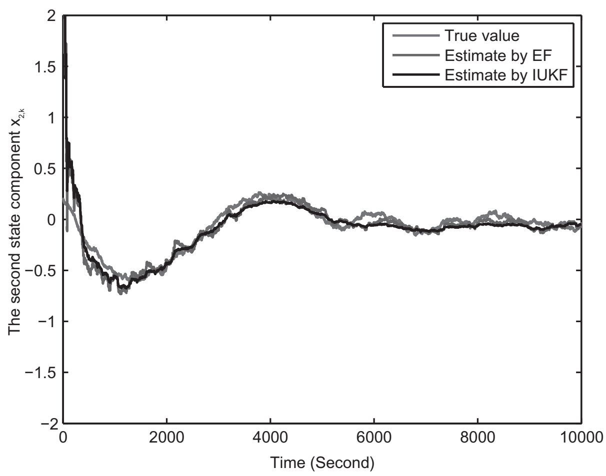

Fig. 1 *Estimate of $x_{2,k}$*

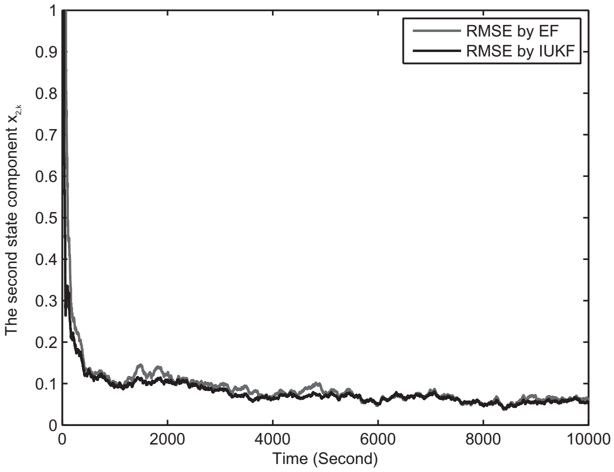

Fig. 2 *RMSE of* $\boldsymbol{x}_{2,k}$

Table 1 Performance metrics for the filters

Filter	Terminal RMSE	Mean computation time (s)
EF	0.535	17.9943
IUKF	0.535	48.4534

effectiveness of this new filtering framework and also indicate that the EF has similar estimation performance with the IUKF; see Fig. 2. However, the IUKF had much longer time (almost three times) than the proposed filter. Therefore the computational overhead of the IUKF and the simplicity of the Jacobian matrix calculations make the proposed EF a better choice for this example.

7 Conclusions

In this paper, the problem of state estimation of non-linear systems with Bernoulli measurement packet losses has been studied by extending the recently proposed suboptimal estimator in [14] to the non-linear case. By generalising the concept of uniform observability, it has been shown that, for certain classes of non-linear systems, there exists a critical value for packet loss rate such that the estimation error covariance matrices are bounded for any initial condition. Furthermore, the behaviour of estimation error has also been investigated and certain conditions for ensuring stochastic stability have been established. Finally, comparative simulations have been conducted to indicate that the EF in this paper is a better choice for this numerical example.

Further research topics include the extension of the proposed filtering method for more general non-linear systems with network-induced phenomena (random sensor delays, quantisation effects, missing measurements) in [18, 19]. Also, it could be of great interest to apply this filtering method to cope with the fault estimation problems discussed in [8, 17].

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