

# Stochastic Loss Minimization for Power Distribution Networks

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**Abstract**—Distribution systems will be critically challenged by reverse power flows and voltage fluctuations due to the integration of distributed renewable generation, demand response, and electric vehicles. Yet the same transformative changes coupled with advances in microelectronics offer new opportunities for reactive power management in distribution grids. In this context and considering the increasing time-variability of distributed generation and demand, a scheme for stochastic loss minimization is developed here. Given uncertain active power injections, a stochastic reactive control algorithm is devised. Leveraging the recent convex relaxation of optimal power flow problems, it is shown that the subgradient of the power losses can be obtained as the Lagrange multiplier of the related second-order cone program (SOCP). Numerical tests on a 47-bus test feeder with high photovoltaic penetration corroborates the power efficiency and voltage profile advantage of the novel stochastic method over its deterministic alternative.

**Index Terms**—Voltage regulation, power loss minimization, stochastic approximation, second-order cone programming, convex relaxation, optimal power flow.

## I. INTRODUCTION

Power distribution systems undergo a transformative change. The integration of photovoltaic (PV) sources, demand response programs, and plug-in hybrid electric vehicles (PHEV), inflict high variability in grid operation. Active power generation and load become highly variable leading to abrupt bus voltage magnitude variations. Reactive power management has been traditionally performed by utility-owned controllable devices, such as tap-changing under load transformers and shunt capacitors [16], [1], [22]. Operational costs, discrete control actions, and slow response times are the factors limiting the use of these devices alone for voltage regulation in distribution system with renewables [12]. Compensating for reactive power along with the control capabilities of distribution generation (DG) units have been advocated as a viable reactive control solution; see e.g., [24], [21]. Albeit current standards call for unit power factor operation [15], the power electronics grid interfaces (inverters) of rooftop PVs can be commanded to provide reactive injections as well [20].

Reactive power compensation via DG units has been an active research area lately. A multi-agent approach has been

proposed in [2]. Voltage regulation was cast as a learning problem in [25]. Control policies based on the approximate *LinDistFlow* model have been developed in [23], [24]; while [9] follows a successive convex approximation approach for voltage regulation. Upon linearizing the power flow equations, a two-layer scheme is proposed in [20]. A consensus-type algorithm is pursued in [5] after approximating power losses as a quadratic function of reactive power injections. Localized reactive power injection updates are reported in [27].

The previous schemes rely on linearizations of the actual system model. Being an instance of the optimal power flow problem (OPF), reactive power management is a non-convex problem, but its convex relaxations have been successful; see [18] for a review. In [10], [11], the OPF for radial distribution grids is surrogated by a second-order cone program (SOCP) building on the *branch flow model* advocated in [3], [4]. Sufficient conditions guaranteeing that the convex relaxation is exact, ensure that solving the relaxed problem is equivalent to solving the original non-convex one [10], [11], [26], [13]. A centralized approach for inverter VAR control using the SOCP relaxation has been devised in [12], while a distributed algorithm based on a semidefinite program relaxation can be found in [17].

The approaches developed so far assume that active injections are precisely known by the distribution energy management system (D-EMS) and remain constant throughout the reactive control interval. However, the D-EMS acquires active injections via noisy measurements, uncertain forecasts, or delayed state estimates. Further, in distribution grids with renewables, active injections fluctuate rapidly unless the control interval is chosen really short. Although commercially available DG solutions are equipped with sensing, control, and communication functionalities (see [20] and references therein); short control intervals come with higher computation and communication demands. In this context, we propose a stochastic framework for power loss minimization by viewing active power injections as random. The *average* rather than the instantaneous power loss is minimized over the controllable reactive injections, while voltages are maintained withing prescribed intervals. After showing that the subgradient of the instantaneous power losses is provided by the Lagrange multipliers of the related SOCP problem, a stochastic approximation algorithm is derived. Numerical tests indicate that the novel

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scheme attains lower losses and a smoother voltage profile over the instantaneous alternative.

*Paper outline.* After the branch flow model is presented in Section II, the problem of stochastic reactive power compensation is formulated in Section III. A stochastic approximation algorithm is developed in Section IV, its performance advantage over an instantaneous reactive control is corroborated in Section V, and conclusions are drawn in Section VI.

## II. SYSTEM MODELING

Consider a power distribution grid consisting of  $N+1$  buses. Due to its radial nature, the grid can be modeled as a tree graph  $\mathcal{T} := (\mathcal{N}_o, \mathcal{L})$ , where  $\mathcal{N}_o := \{0, \dots, N\}$  and  $|\mathcal{L}| = N$ . The tree is rooted at the substation bus indexed by  $i = 0$ . For every bus  $i \in \mathcal{N}_o$ , let  $v_i$  be its squared voltage magnitude,  $V_i = \sqrt{v_i}e^{j\theta_i}$  its complex voltage, and  $s_i = p_i + jq_i$  the complex power injected into bus  $i$ . Since every non-root bus  $i \in \mathcal{N} := \{1, \dots, N\}$  has a unique parent bus denoted by  $\pi_i$ , the line  $(\pi_i, i)$  feeding bus  $i$  can be simply indexed by  $i$  with  $i \in \mathcal{L} := \{1, \dots, N\}$ ; see Fig. 1. If  $z_i = r_i + jx_i$  denotes the impedance of line  $i$ , and  $I_i = \sqrt{l_i}e^{j\phi_i}$  the complex current flowing from  $\pi_i$  to  $i$  with  $l_i$  being the squared current magnitude, then Ohm's law dictates that

$$V_{\pi_i} - V_i = I_i z_i \quad (1)$$

for all lines  $i \in \mathcal{L}$ . The complex power flow on line  $i$  can be expressed as

$$S_i := P_i + jQ_i = V_{\pi_i} I_i^* \quad (2)$$

for every line  $i \in \mathcal{L}$ . The power flow received at the  $i$ -side of line  $(\pi_i, i)$  can be shown to be  $S_i - z_i l_i$ . Finally, conservation of energy implies that per bus  $i \in \mathcal{N}$

$$s_i = \sum_{k \in \mathcal{C}_i} S_k - (S_i - z_i l_i) \quad (3)$$

where  $\mathcal{C}_i := \{k \in \mathcal{N} : \pi_k = i\}$  is the set of the children nodes for bus  $i$ . Equations (1)-(3) constitute a complete AC model for the grid. However, to derive efficient algorithms, the so termed *branch flow model* will be utilized here instead [3], [4]. The branch flow model ignores voltage phases  $\{\theta_i\}_{i \in \mathcal{N}_o}$  and current phases  $\{\phi_i\}_{i \in \mathcal{L}}$  by squaring (1) and (2). The obtained model is described by the equations ( $i = 1, \dots, N$ )

$$p_i = \sum_{k \in \mathcal{C}_i} P_k - (P_i - r_i l_i) \quad (4)$$

$$q_i = \sum_{k \in \mathcal{C}_i} Q_k - (Q_i - x_i l_i) \quad (5)$$

$$v_i = v_{\pi_i} + (r_i^2 + x_i^2)l_i - 2(r_i P_i + x_i Q_i) \quad (6)$$

$$l_i = \frac{P_i^2 + Q_i^2}{v_{\pi_i}} \quad (7)$$

and recall that  $l_i$  and  $v_i$  are the squared amplitudes for the line  $i$ 's current and bus  $i$ 's voltage, accordingly. Equations (4)-(6) are accompanied with the initial conditions  $v_0 = 1$ ,  $p_0 = \sum_{k \in \mathcal{C}_0} P_k$ , and  $q_0 = \sum_{k \in \mathcal{C}_0} Q_k$ .

Collect all nodal quantities related to non-root buses in vectors  $\mathbf{p} := [p_1 \dots p_N]'$ ,  $\mathbf{q} := [q_1 \dots q_N]'$ , and  $\mathbf{v} :=$

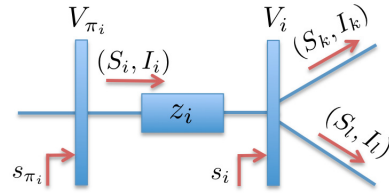


Fig. 1. Bus  $i$  and its unique parent  $\pi_i$  are connected via line  $i$ . Variables  $(S_i, I_i, z_i)$  represent the complex power, current, and impedance on line  $i$ , while  $(s_i, V_i)$  the complex power injection and voltage on bus  $i$ , respectively.

$[v_1 \dots v_N]'$ , where the prime  $'$  denotes vector transposition. Likewise, for line quantities define  $\mathbf{P} := [P_1 \dots P_N]'$ ,  $\mathbf{Q} := [Q_1 \dots Q_N]'$ , and  $\mathbf{l} := [l_1 \dots l_N]'$ . Bus voltage magnitudes are allowed to lie within a prespecified range, i.e.,  $v_i \in [\underline{v}_i, \bar{v}_i]$  for every  $i \in \mathcal{N}$ . Upon setting  $\underline{\mathbf{v}} := [\underline{v}_1 \dots \underline{v}_N]'$  and  $\bar{\mathbf{v}} := [\bar{v}_1 \dots \bar{v}_N]'$ , voltage regulation constraints can be compactly expressed as

$$\mathbf{v} \in \mathcal{V} := \{\mathbf{v} : \underline{\mathbf{v}} \preceq \mathbf{v} \preceq \bar{\mathbf{v}}\} \quad (8)$$

where  $\preceq$  denotes an entry-wise vector inequality. Building on the branch flow model of (4)-(8), our novel reactive control scheme is formulated next.

## III. PROBLEM STATEMENT

The active and reactive power injection at bus  $i$  can be decomposed into its generation and consumption components, that is  $p_i = p_i^g - p_i^c$  and  $q_i = q_i^g - q_i^c$ . For a load bus, there is no generation ( $p_i^g = q_i^g = 0$ ), the consumed active power is  $p_i^c \geq 0$ , and its reactive power  $q_i^c \geq 0$  is typically related to  $p_i^c$  through a constant power factor. When it comes to a DG bus (e.g., a house or an industrial facility equipped with PVs or a wind turbine), both active and reactive powers could be generated as well. For a bus hosting a shunt capacitor,  $p_i = 0$  and  $q_i^g > 0$ .

The grid operation can be divided into short time periods indexed by  $t$ , over which electrical quantities can be initially assumed to remain constant. If  $(\mathbf{p}_t, \mathbf{q}_t)$  are the active and reactive power injections in all but the root buses during the control interval  $t$ , the related power loss is expressed as

$$f(\mathbf{p}_t, \mathbf{q}_t) := \sum_{i=0}^N p_{i,t} = \sum_{i=1}^N r_i l_{i,t} \quad (9)$$

where the second equality follows readily from (4).

Given the active power injections  $\mathbf{p}_t$  and the corresponding reactive load demands  $\mathbf{q}_t^c$ , conventional reactive power management aims at choosing  $\mathbf{q}_t^g$  so that power losses are minimized and voltages are maintained within  $\mathcal{V}$ . Minimizing power losses on a per time  $t$  basis would be the optimal policy under two conditions:  $(\mathbf{p}_t, \mathbf{q}_t^c)$  are precisely known and remain constant over the operation interval  $t$ . Unfortunately, such operating conditions will be hardly met in future distribution grids: High penetration of renewables will incur time-varying active and reactive power injections. Moreover, in low-inertia microgrids, the lack of droop controllers would

further challenge voltage regulation. But even in a slow time-varying setup where the actual  $(\mathbf{p}_t, \mathbf{q}_t^c)$  are relatively constant over the time period  $t$ , the D-EMS will only have available noise-contaminated observations (e.g., either via direct measurements or through the delayed output of the power system state estimator).

Based on these data, a stochastic optimization approach is pursued here. Active and reactive power injections  $(\mathbf{p}_t, \mathbf{q}_t^c)$  are modeled as *random variables* drawn independently across time from a probability density function. The control task is to minimize the *average* grid power loss  $\mathbb{E}_{(\mathbf{p}_t, \mathbf{q}_t^c)}[f(\mathbf{p}_t, \mathbf{q}_t)]$  by properly setting the control variables  $\mathbf{q}^g$ . Recalling that  $\mathbf{q}_t = \mathbf{q}^g - \mathbf{q}_t^c$ , define for notational brevity

$$f_t(\mathbf{q}^g) := f(\mathbf{p}_t, \mathbf{q}^g - \mathbf{q}_t^c).$$

A stochastic power loss minimization approach entails finding

$$\hat{\mathbf{q}}^g := \arg \min_{\mathbf{q}^g \in \mathcal{Q}} \mathbb{E}[f_t(\mathbf{q}^g)] \quad (10)$$

where  $\mathcal{Q}$  is the feasible region for  $\mathbf{q}^g$  that is assumed to be convex. An algorithm for solving (10) is presented next.

#### IV. POWER LOSS MINIMIZATION SCHEME

Assuming  $\mathbb{E}[f_t(\mathbf{q}^g)]$  to be convex, standard projected subgradient descent can solve (10) by iteratively updating

$$\hat{\mathbf{q}}_{k+1}^g := [\hat{\mathbf{q}}_k^g - \mu_1 \mathbf{g}_k]_{\mathcal{Q}} \quad (11)$$

where  $\mathbf{g}_k$  belongs to the subdifferential  $\partial \mathbb{E}[f_t(\mathbf{q}_k^g)]$  [6], the operator  $[\cdot]_{\mathcal{Q}}$  stands for the projection onto the feasible set  $\mathcal{Q}$ , and  $\mu_1 > 0$  is a sufficiently small step size. When the feasible set is a box, i.e.,  $\mathcal{Q} = [\underline{\mathbf{q}}^g, \bar{\mathbf{q}}^g]$ , the  $i$ -th entry of the projected vector  $\mathbf{z}_{\mathcal{Q}} := [\mathbf{z}]_{\mathcal{Q}}$  is simply

$$z_{\mathcal{Q},i} = \begin{cases} \bar{q}_i^g, & z_i > \bar{q}_i^g \\ z_i, & z_i \in [q_i^g, \bar{q}_i^g] \\ \underline{q}_i^g, & z_i < \underline{q}_i^g \end{cases}. \quad (12)$$

Even if the joint probability density function of  $(\mathbf{p}_t, \mathbf{q}_t^c)$  were known, evaluating the expectation in (11) would be non-trivial. To practically address these challenges, a stochastic approximation approach is advocated next.

Under the stochastic approximation framework [19],  $\hat{\mathbf{q}}^g$  is updated as soon as a new datum  $(\mathbf{p}_t, \mathbf{q}_t^c)$  becomes available. In other words, the update index  $k$  coincides with the operation time  $t$ . Even more critically though, the update at time  $t$  depends only on a subgradient  $\mathbf{g}_t$  belonging to the subdifferential  $\partial f_t(\hat{\mathbf{q}}^g)$  evaluated at  $\hat{\mathbf{q}}_t^g$ . Precisely, reactive power at DG units can be updated as

$$\hat{\mathbf{q}}_{t+1}^g := [\hat{\mathbf{q}}_t^g - \mu_2 \mathbf{g}_t]_{\mathcal{Q}}. \quad (13)$$

Implementing the reactive control scheme of (13) involves finding  $\mathbf{g}_t \in \partial f_t(\hat{\mathbf{q}}_t^g)$  and choosing the step size  $\mu_2 > 0$ .

To find a subgradient, recall that  $f_t(\hat{\mathbf{q}}_t^g)$  is the loss dissipated on distribution lines when injecting  $(\mathbf{p}_t, \hat{\mathbf{q}}_t^g - \mathbf{q}_t^c)$  into the grid. Provided that  $(\mathbf{p}_t, \hat{\mathbf{q}}_t^g - \mathbf{q}_t^c)$  is feasible, function  $f_t(\hat{\mathbf{q}}_t^g)$  depends on the underlying current magnitudes [cf. (9)]. Given power injections  $(\mathbf{p}, \mathbf{q})$ , if the substation can provide sufficient  $(p_0, q_0)$

and voltages are maintained in the prescribed operation range, meaning  $\mathbf{v} \in \mathcal{V}$ ; there exists a unique grid operating point  $(p_0, q_0, \mathbf{p}, \mathbf{q}, \mathbf{v}, \mathbf{P}, \mathbf{Q}, \mathbf{l})$  satisfying the nonlinear equations in (4)-(7) [8]. If the equalities in (7) are relaxed to inequalities, then  $(p_0, q_0, \mathbf{v}, \mathbf{P}, \mathbf{Q}, \mathbf{l})$  lies in a convex set [10], [11], [12]. This convex set is represented by the linear equalities (4)-(6), the set  $\mathcal{V}$ , and the second-order cone constraints

$$\{P_i^2 + Q_i^2 \leq l_i v_{\pi_i}\}_{i \in \mathcal{L}}. l_i \geq \frac{P_i^2 + Q_i^2}{v_{\pi_i}}, \quad i \in \mathcal{L}$$

for all  $i \in \mathcal{L}$ . Under different technical conditions (for example, see [10], [11], [26], [13]), the minimizer of the convex optimization problem

$$f(\mathbf{p}, \mathbf{q}) = \min_{\substack{\mathbf{P}, \mathbf{Q}, \mathbf{l} \\ p_0, q_0, \mathbf{v}}} \sum_{i=1}^L r_i l_i \quad (14a)$$

$$\text{s.to } p_i = \sum_{k \in \mathcal{C}_i} P_k - (P_i - r_i l_i), \quad i \in \mathcal{N} \quad (14b)$$

$$q_i = \sum_{k \in \mathcal{C}_i} Q_k - (Q_i - x_i l_i), \quad i \in \mathcal{N} \quad (14c)$$

$$v_i = v_{\pi_i} + (r_i^2 + x_i^2) l_i - 2(r_i P_i + x_i Q_i), \quad i \in \mathcal{N} \quad (14d)$$

$$l_i \geq \frac{P_i^2 + Q_i^2}{v_{\pi_i}}, \quad i \in \mathcal{L} \quad (14e)$$

$$\mathbf{v} \in \mathcal{V} \quad (14f)$$

satisfies the SOCP constraints in (14e) with equality. When this occurs, the convex relaxation is said to be *exact*. In that case, the  $(p_0, q_0, \mathbf{P}, \mathbf{Q}, \mathbf{l}, \mathbf{v})$  minimizing (14) corresponds to the underlying operating point, and thus  $f(\mathbf{p}, \mathbf{q})$  is the actual power loss. Henceforth, it will be assumed that:

**(a1)**  $(\mathbf{p}, \mathbf{q})$  is feasible; and

**(a2)** the convex relaxation in (14) is exact.

Assumptions (a1)-(a2) are verified during our numerical tests.

Since  $\mathbf{q} = \mathbf{q}^g - \mathbf{q}_t^c$  and  $\mathbf{q}_t^c$  is fixed, finding a subgradient of  $f_t(\mathbf{q}_t^g)$  boils down to finding a subgradient of  $f_t(\mathbf{q})$ . The important observation here is that the function  $f_t(\mathbf{q})$  can be alternatively expressed as

$$f_t(\mathbf{q}) = \min_{\mathbf{z}} \mathbf{c}_t' \mathbf{z} \quad (15a)$$

$$\text{s.to } \mathbf{z} \in \mathcal{Z}_t \quad (15b)$$

$$\mathbf{A} \mathbf{z} = \mathbf{q} \quad (15c)$$

after identifying  $\mathbf{z} := [p_0 \ q_0 \ \mathbf{P}' \ \mathbf{Q}' \ \mathbf{l}' \ \mathbf{v}']'$  and  $\mathbf{c}_t := [\mathbf{0}'_{2L+2} \ \mathbf{r}' \ \mathbf{0}'_N]'$ , where  $\mathbf{0}_N$  is the all-zeros vector of dimension  $N$  and  $\mathbf{r}$  is the  $L \times 1$  vector of line resistances. The constraint (15c) models the linear equalities in (14c), and all other constraints of (14) are abstractly denoted by the convex set  $\mathcal{Z}_t$  in (15b). Based on this alternative expression of  $f_t(\mathbf{q})$ , the next two results proved in the Appendix can be obtained.

**Lemma 1.** *Under (a2), function  $\mathbb{E}[f_t(\mathbf{q}^g)]$  is convex on  $\mathcal{Q}$ .*

**Lemma 2.** *If the convex relaxation in (14) is exact and strong duality holds, then the negative vector of the Lagrange*

TABLE I  
LINE DATA FOR THE 47-BUS DISTRIBUTION FEEDER

From Bus	To Bus	$r_i$ [ $\Omega$ ]	$x_i$ [ $\Omega$ ]	From Bus	To Bus	$r_i$ [ $\Omega$ ]	$x_i$ [ $\Omega$ ]
1	2	0.259	0.808	11	12	0.076	0.046
2	3	0.031	0.092	11	47	0.031	0.015
2	13	0	0	15	16	0.107	0.015
3	4	0.046	0.092	15	18	0.046	0.015
3	14	0.092	0.031	16	17	0	0
3	15	0.214	0.046	18	19	0	0
4	5	0.107	0.183	20	21	0.122	0.092
4	20	0.336	0.061	20	25	0.214	0.046
5	6	0.015	0.031	21	22	0.198	0.046
5	26	0.061	0.015	21	24	0	0
6	7	0.031	0.046	22	23	0	0
6	27	0.168	0.061	27	28	0.107	0.031
7	8	0.015	0.015	27	31	0.046	0.015
7	32	0.076	0.015	28	29	0.107	0.031
8	9	0.031	0.031	29	30	0.061	0.015
8	35	0.076	0.015	32	33	0.046	0.015
8	39	0.244	0.046	33	34	0.031	0
8	40	0.046	0.015	35	36	0.076	0.015
8	41	0.107	0.031	35	37	0.076	0.046
9	10	0.015	0.015	35	38	0.107	0.015
9	42	0.153	0.046	42	43	0.061	0.015
10	11	0.107	0.076	43	44	0.061	0.015
10	46	0.229	0.122	43	45	0.061	0.015

TABLE II  
BUS DATA FOR THE 47-BUS DISTRIBUTION FEEDER

Load data				Shunt capacitors	
Bus	Peak [MVA]	Bus	Peak [MVA]	Bus	Nameplate capacity [MVAR]
1	30	31	0.07	1	6.0
11	0.67	32	0.13	3	1.20
12	0.45	33	0.27	37	1.80
14	0.89	34	0.20	47	1.80
16	0.07	36	0.27	<b>PV generators</b>	
18	0.67	38	0.45	Bus	Nameplate capacity [MW]
21	0.45	39	1.34	13	1.50
22	2.23	40	0.13	17	0.40
25	0.45	41	0.67	19	1.50
26	0.20	42	0.13	23	1.00
28	0.13	44	0.45	24	2.00
29	0.13	45	0.20		
30	0.20	46	0.45		
Base information					
Voltage base $V_B = 12.35KV$					
Power base $P_B = 1000KVA$					

multipliers corresponding to the linear constraints in (15c)  $-\lambda$  belongs to  $\partial f_t(\mathbf{q})$ .

Lemma 2 asserts that for the update in (13), it suffices to find the Lagrange multipliers corresponding to (14e) while solving (14) for  $\mathbf{p} = \mathbf{p}_t^g - \mathbf{p}_t^c$  and  $\mathbf{q} = \hat{\mathbf{q}}_t^g - \mathbf{q}_t^c$ .

## V. NUMERICAL TESTS

The merits of the novel stochastic reactive power compensation scheme over its deterministic counterpart is evaluated using the 47-bus distribution feeder shown in Fig. 2. This circuit is a simplified model of an industrial distribution feeder of South California Edison with high penetration of

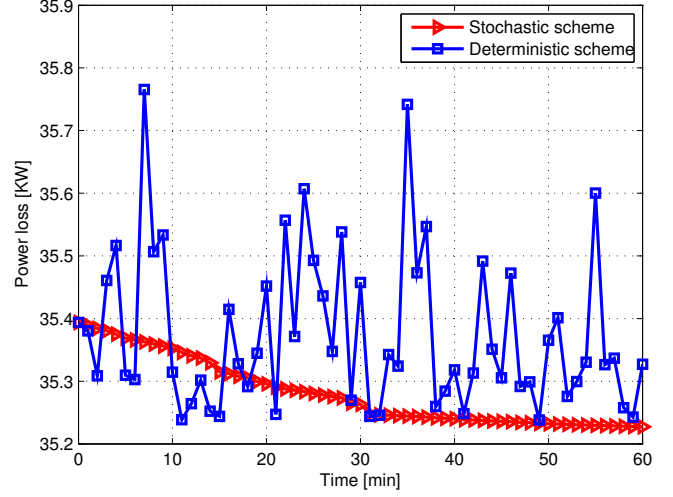


Fig. 3. Actual power losses for the noisy ( $\mathbf{p}^g, \mathbf{p}^c, \mathbf{q}^c$ ) model.

renewables represented by 5 PV systems [12]. Line impedance data are listed in Table I, while bus data including the peak load demand, the nameplate capacity of PV generators and shunt capacitors, as well as the system base information are presented in Table II. All loads are assumed to operate at a constant power factor of 0.8.

A control period of 1 hour is considered and it is divided in 60 grid operation intervals. For each interval, the D-EMS collects active and reactive injections from load buses, as well as active injections from DG buses. Without loss of generality and to model higher renewable integration, shunt capacitors are considered controllable DG units as well. The reactive power injections from DG units are determined: (i) by solving (14) on a per-minute basis (henceforth deterministic scheme); and (ii) via the devised stochastic scheme. The step size  $\mu_2$  was empirically set to 1. It is worth mentioning that all SOCP relaxations were feasible and exact.

In our first experiment, injections ( $\mathbf{p}^g, \mathbf{p}^c, \mathbf{q}^c$ ) are considered fixed throughout the 1-hour interval. At each minute though, the D-EMS observes a noise-corrupted version of  $\mathbf{p}^g$  as  $\mathbf{p}_t^g = \mathbf{p}^g + \boldsymbol{\eta}_t$ , where the entries of  $\boldsymbol{\eta}_t$  are independently and uniformly drawn from  $[-0.05, 0.05]$ . Likewise, noisy readings are collected for  $\mathbf{p}^c$  and  $\mathbf{q}^c$ . Even though  $\hat{\mathbf{q}}_t^g$  is decided upon the aforementioned noise-corrupted injections, the power loss experienced is apparently  $f(\mathbf{p}^g - \mathbf{p}^c, \hat{\mathbf{q}}_t^g - \mathbf{q}^c)$ .

Figure 3 depicts the actual power losses for the two reactive control schemes, and verifies the efficiency gain of the stochastic one. The corresponding voltage fluctuations are depicted in Fig. 4, where it is corroborated that the proposed algorithm yields smoother voltage profiles. Figure 5 shows the actual power losses experienced by the two control schemes, after they have been averaged over 30 independent noise realizations. The deterministic scheme suffers an average value of 35.35KW, whereas the stochastic one converges to an average value of 35.26KW.

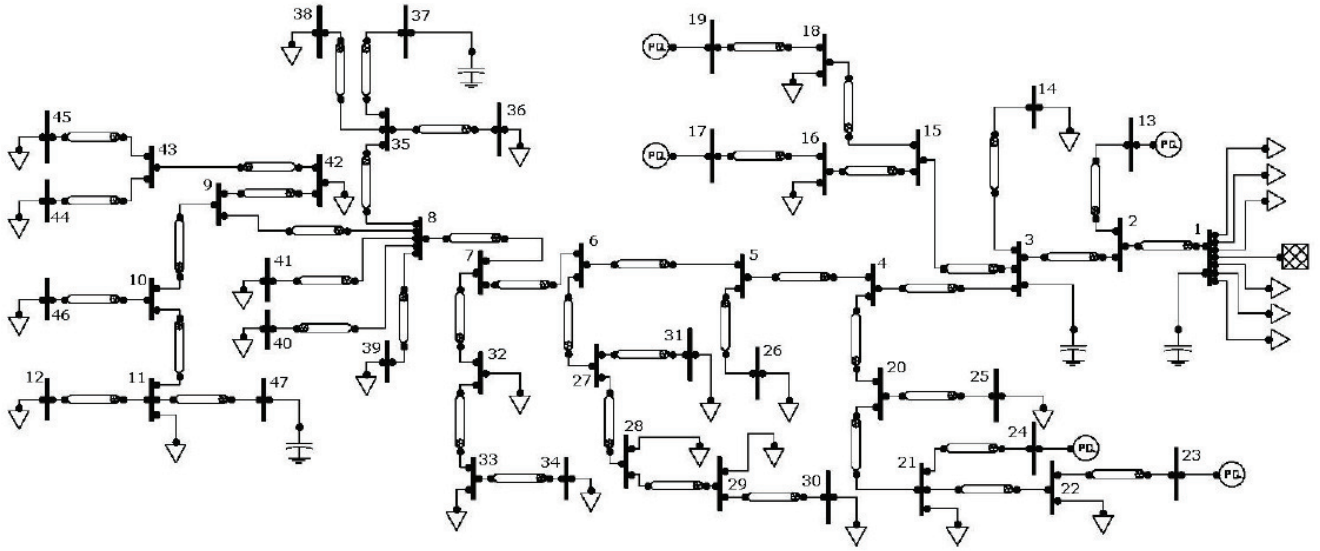


Fig. 2. Schematic diagram of a 47-bus distribution feeder with high penetration of photovoltaics. The 6 loads attached to the substation bus (indexed as bus 1 here) model additional feeders.

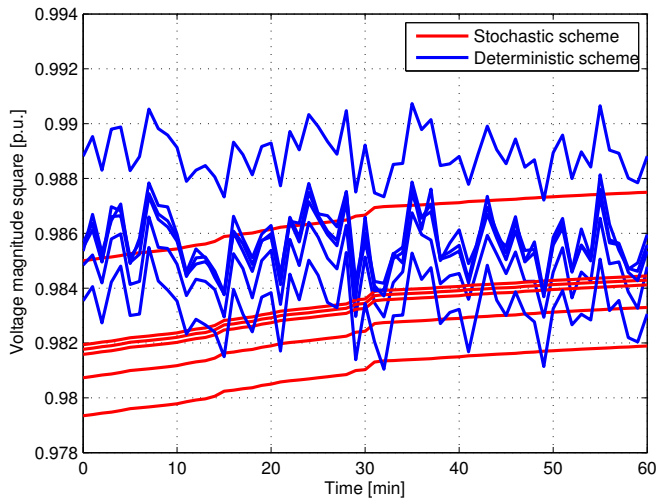


Fig. 4. Voltage magnitude across buses No. 5, 12, 29, 34, 39, and 45, for the noisy  $(\mathbf{p}^g, \mathbf{p}^c, \mathbf{q}^c)$  model.

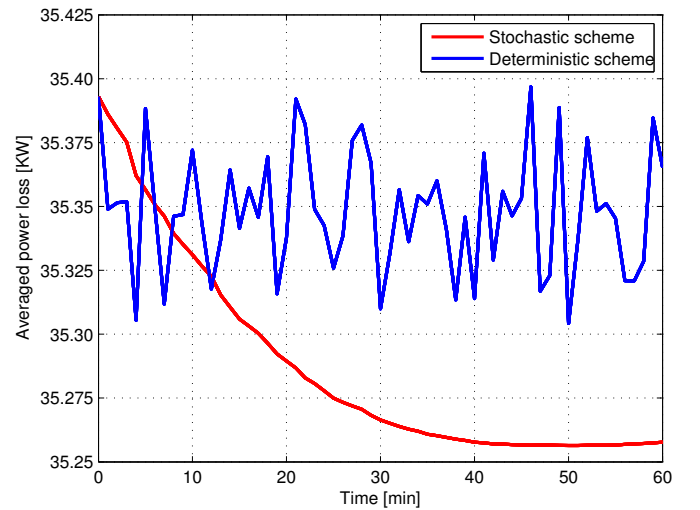


Fig. 5. Actual power losses averaged over 30 independent realizations for the noisy  $(\mathbf{p}^g, \mathbf{p}^c, \mathbf{q}^c)$  model.

Under a more realistic setup, problem parameters  $\mathbf{p}^g$ ,  $\mathbf{p}^c$ , and  $\mathbf{q}^c$ , are assumed to follow a random walk model. The injection at minute  $t + 1$  equals the value at minute  $t$  plus a zero-mean white Gaussian noise having standard deviation 0.004. This scenario models inevitable sensing, computation, and communication delays: reactive compensation  $\hat{\mathbf{q}}_t^g$  is based on  $(\mathbf{p}_t^g, \mathbf{p}_t^c, \mathbf{q}_t^c)$ ; yet the grid experiences losses  $f(\mathbf{p}_{t+1}^g - \mathbf{p}_{t+1}^c, \hat{\mathbf{q}}_t^g - \mathbf{q}_{t+1}^c)$ . The power losses averaged over 30 independent realizations are shown in Figure 6 and demonstrate that the stochastic algorithm indeed outperforms the deterministic reactive power compensation scheme.

## VI. CONCLUDING REMARKS

The problem of power loss minimization has been considered in this work. Uncertainty and variability in the distribution system state as well as sensing, communication, and control delays, call for stochastic solutions. Building on the recent advances on the convex relaxations of the OPF task, a novel stochastic approximation scheme was developed. Reactive injections were updated according to the Lagrange multipliers of the associated SOCP problem. Incorporating voltage regulating transformers, choosing the step size, assessing the effects of grid topology, and performing the updates more efficiently, are some of our current research directions.



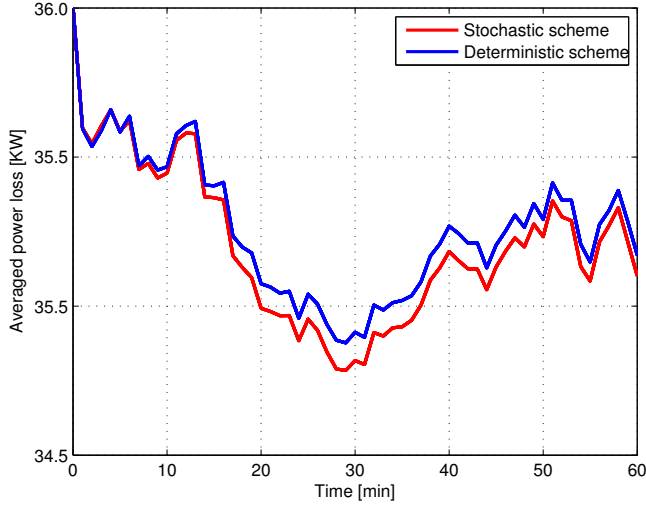


Fig. 6. Average power losses for the random walk model.

## APPENDIX

*Proof of Lemma 1:* It suffices to show that  $f_t(\mathbf{q}^g)$  is convex on  $\mathcal{Q}$  for every  $t$ . The latter follows easily since under (a2), the equality in (15a) holds and the function in the right-hand side of (15a) is known to be convex (its epigraph is a convex set [6, Sec. 5.6.1]). ■

*Proof of Lemma 2:* The exactness of (14) implies the feasibility of (15) and the finiteness of the optimal value  $f_t(\mathbf{q})$ . By dualizing only the linear constraint, the Lagrangian function of (15) is

$$L(\mathbf{z}; \boldsymbol{\lambda}) := \mathbf{c}'_t \mathbf{z} + \boldsymbol{\lambda}' (\mathbf{A} \mathbf{z} - \mathbf{q})$$

and the corresponding dual function is

$$g(\boldsymbol{\lambda}) = \min_{\mathbf{z} \in \mathcal{Z}_t} L(\mathbf{z}; \boldsymbol{\lambda}).$$

Let  $\mathbf{z}_1$  and  $\boldsymbol{\lambda}$  be optimal primal and dual variables for  $f_t(\mathbf{q})$ , respectively. Further, let  $\mathbf{z}_2$  be the optimal primal variable for  $f_t(\mathbf{q} + \mathbf{v})$  for some  $\mathbf{v}$ , which means that  $\mathbf{z}_2 \in \mathcal{Z}_t$  and  $\mathbf{A} \mathbf{z}_2 = \mathbf{q} + \mathbf{v}$ . It holds that

$$\mathbf{c}'_t \mathbf{z}_1 = g(\boldsymbol{\lambda}) \leq \mathbf{c}'_t \mathbf{z}_2 + \boldsymbol{\lambda}' (\mathbf{A} \mathbf{z}_2 - \mathbf{q}) = \mathbf{c}'_t \mathbf{z}_2 + \boldsymbol{\lambda}' \mathbf{v} \quad (16)$$

where the first equality comes from strong duality; the inequality holds because  $g(\boldsymbol{\lambda}) \leq L(\mathbf{z}; \boldsymbol{\lambda})$  for every  $\mathbf{z} \in \mathcal{Z}_t$ , and hence for  $\mathbf{z}_2$ ; and the last equality holds because  $\mathbf{A} \mathbf{z}_2 = \mathbf{q} + \mathbf{v}$ . The inequality in (16) implies that  $f_t(\mathbf{q} + \mathbf{v}) \geq f_t(\mathbf{q}) - \boldsymbol{\lambda}' \mathbf{v}$  for any  $\mathbf{v}$ , proving that  $-\boldsymbol{\lambda} \in \partial f_t(\mathbf{q})$ ; see e.g., [6]. ■

## REFERENCES

- [1] R. Baldick and F. Wu, "Efficient integer optimization algorithms for optimal coordination of capacitors and regulators," *IEEE Trans. Power Syst.*, vol. 5, no. 3, pp. 805–812, Aug. 1990.
- [2] M. Baran and I. El-Markabi, "A multiagent-based dispatching scheme for distributed generators for voltage support on distribution feeders," *IEEE Trans. Power Syst.*, vol. 22, no. 1, pp. 52–59, Feb. 2007.
- [3] M. Baran and F. Wu, "Optimal capacitor placement on radial distribution systems," *IEEE Trans. Power Syst.*, vol. 4, no. 1, pp. 725–734, Jan. 1989.

- [4] —, "Optimal sizing of capacitors placed on a radial distribution system," *IEEE Trans. Power Syst.*, vol. 4, no. 1, pp. 735–743, Jan. 1989.
- [5] S. Bolognani, R. Carli, G. Cavraro, and S. Zampieri, "A distributed control strategy for optimal reactive power flow with power constraints," in *Proc. IEEE Conf. on Decision and Control*, Florence, Italy, Dec. 2013.
- [6] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY: Cambridge University Press, 2004.
- [7] P. M. S. Carvalho, P. F. Correia, and L. A. Ferreira, "Distributed reactive power generation control for voltage rise mitigation in distribution networks," *IEEE Trans. Power Syst.*, vol. 23, no. 2, pp. 766–772, May 2008.
- [8] H.-D. Chiang and M. Baran, "On the existence and uniqueness of load flow solution for radial distribution power networks," *IEEE Trans. Circuits Syst.*, vol. 37, no. 3, pp. 410–416, Mar. 1990.
- [9] S. Deshmukh, B. Natarajan, and A. Pahwa, "Voltage/VAR control in distribution networks via reactive power injection through distributed generators," *IEEE Trans. Smart Grid*, vol. 3, no. 3, pp. 1226–1234, Sep. 2012.
- [10] M. Farivar and S. Low, "Branch flow model: Relaxations and convexification — Part I," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2554–2564, Aug. 2013.
- [11] —, "Branch flow model: Relaxations and convexification — Part II," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2565–2572, Aug. 2013.
- [12] M. Farivar, R. Neal, C. Clarke, and S. Low, "Optimal inverter VAR control in distribution systems with high PV penetration," in *Proc. IEEE Power & Energy Society General Meeting*, San Diego, CA, Jul. 2012, pp. 1–7.
- [13] L. Gan, N. Li, U. Topcu, and S. Low, "On the exactness of convex relaxation for optimal power flow in tree networks," in *Proc. IEEE Conf. on Decision and Control*, Maui, HI, Dec. 2012, pp. 465–471.
- [14] C. Guille and G. Gross, "A conceptual framework for the vehicle-to-grid (V2G) implementation," *Energy Policy*, vol. 37, no. 11, pp. 4379–4390, Nov. 2009.
- [15] *IEEE 1547 Standard for Interconnecting Distributed Resources with Electric Power Systems*, IEEE Std., 2011. [Online]. Available: [http://grouper.ieee.org/groups/scc21/1547/1547\\_index.html](http://grouper.ieee.org/groups/scc21/1547/1547_index.html)
- [16] W. H. Kersting, *Distribution System Modeling and Analysis*. New York, NY: CRC Press, 2001.
- [17] A. Lam, B. Zhang, and D. Tse, "Distributed algorithms for optimal power flow problem," in *Proc. IEEE Conf. on Decision and Control*, Maui, HI, Dec. 2012, pp. 430–437.
- [18] S. Low, "Convex relaxation of optimal power flow — Part II: exactness," *IEEE Trans. Control of Network Systems*, no. 99, pp. 1–1, 2014.
- [19] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro, "Robust stochastic approximation approach to stochastic programming," *SIAM J. Optim.*, vol. 19, no. 4, pp. 1574–1609, 2009.
- [20] B. Robbins, C. Hadjicostis, and A. Dominguez-Garcia, "A two-stage distributed architecture for voltage control in power distribution systems," *IEEE Trans. Power Syst.*, vol. 28, no. 2, pp. 1470–1482, May 2013.
- [21] K. Rogers, R. Klump, H. Khurana, A. Aquino-Lugo, and T. Overbye, "An authenticated control framework for distributed voltage support on the smart grid," *IEEE Trans. Smart Grid*, vol. 1, no. 1, pp. 40–47, Jun. 2010.
- [22] T. Senjyu, Y. Miyazato, A. Yona, N. Urasaki, and T. Funabashi, "Optimal distribution voltage control and coordination with distributed generation," *IEEE Trans. Power Delivery*, vol. 23, no. 2, pp. 1236–1242, Apr. 2008.
- [23] K. Turitsyn, P. Sulc, S. Backhaus, and M. Chertkov, "Distributed control of reactive power flow in a radial distribution circuit with high photovoltaic penetration," in *Proc. IEEE Power & Energy Society General Meeting*, Minneapolis, MN, Jul. 2010, pp. 1–6.
- [24] —, "Options for control of reactive power by distributed photovoltaic generators," *Proc. IEEE*, vol. 99, no. 6, pp. 1063–1073, Jun. 2011.
- [25] D. Villacci, G. Bontempi, and A. Vaccaro, "An adaptive local learning-based methodology for voltage regulation in distribution networks with dispersed generation," *IEEE Trans. Power Syst.*, vol. 21, no. 3, pp. 1131–1140, Aug. 2006.
- [26] B. Zhang and D. Tse, "Geometry of injection regions of power networks," *IEEE Trans. Power Syst.*, vol. 28, no. 2, pp. 788–797, Feb. 2013.
- [27] B. Zhang, A. Dominguez-Garcia, and D. Tse, "A local control approach to voltage regulation in distribution networks," in *Proc. North American Power Symposium*, Manhattan, KS, Sep. 2013, pp. 1–6.